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Self-gravity of brane worlds: a new hierarchy twist

Christos Charmousis

Institute for Fundamental Theory, Department of Physics, University of Florida Gainesville FL 32611-8440, USA
E-mail: charmousis@phys.ufl.edu

Roberto Emparan*

Departamento de Física Teórica, Universidad del País Vasco Apdo. 644, E-48080 Bilbao, Spain
E-mail: wtpemgar@lg.ehu.es

Ruth Gregory

Centre for Particle Theory, Durham University South Road, Durham, DH1 3LE, U.K.
E-mail: R.A.W.Gregory@durham.ac.uk

Abstract: We examine the inclusion of brane self-gravity in brane-world scenarios with three or more compact extra dimensions. If the brane is a thin, localized one, then we find that the geometry in its vicinity is warped in such a way that gravity on the brane can become very weak, independently of the volume of the extra dimensions. As a consequence, self-gravity can make the brane structure enter into the determination of the hierarchy between the Planck scale and a lower fundamental scale. In an extreme case, one can obtain a novel reformulation of the hierarchy problem in brane worlds, without the need for large-size extra dimensions; the hierarchy would be generated when the ratio between the scales of brane tension and brane thickness is large. In a sense, such a scenario is half-way between the one of Arkani-Hamed et al. (ADD) (although with TeV-mass Kaluza-Klein states) and that of Randall and Sundrum (RS1) (but with only a TeV brane, and of positive tension). We discuss in detail the propagation of fields in the background of this geometry, and find that no problems appear even if the brane is taken to be very thin. We also discuss the presence of black branes and black holes in this setting, and the possibility of having a Planck brane.

Keywords: Extra Large Dimensions, p-branes, Black Holes.

*From Jan 2001, also at Theory Division, CERN, CH-1211, Geneva 23, Switzerland. E-mail: roberto.emparan@cern.ch

JHEP05(2001)026

Contents

1. Introduction

The idea that spacetime may have more than four dimensions, with some of them being macroscopically large, has recently become an attractive scenario where many a model has been proposed. The common defining features are, on the one hand, the presence of a three-brane where the Standard Model is confined, and on the other hand, a bulk where gravity propagates. Other fields may live in either brane or bulk, as long as their current undetectability is consistent with experimental bounds.

The idea dates back at least to [1], but, at the risk of oversimplification, we may identify two main lines in the recent developments: one following the works of Arkani-Hamed et al (ADD) [2], and the other that of Randall and Sundrum (RS) [3, 4]. Each of these seeks a novel, geometrical, resolution of the hierarchy problem — the volume (effective or real) of the additional dimensions providing the hierarchy between the Planck and standard model scales. Works under the latter, RS, class typically consider the brane-world to be a defect of low codimension, say one or two, i.e. a 'domain wall' (as in the original RS scenario) or a 'cosmic string', [5, 6] and explicitly include the self-gravity (backreaction) of the brane into the bulk of, typically, an anti-de Sitter spacetime. By contrast, the scenario proposed by ADD neglects the self-gravity of the brane, and regards the number of extra dimensions, n, as essentially an adjustable parameter. In these cases, typically, the extra dimensions have to be compactified in order to recover the four-dimensional law of gravity at least down to distances above the experimental bounds [7]. It is quite possible that in the near future such scenarios with $n < 3$ extra dimensions will be ruled out.

There are in fact good reasons why the bending of spacetime produced by the brane is dealt with so differently in these two kinds of scenarios. Typically, a localised defect of codimension one or two manifests its self-gravity in global features of its spacetime; this is because the intrinsic spatial directions of the defect do not participate in the gravitational interaction, and the symmetries of the defect therefore mean that gravity is effectively propagating only in the directions orthogonal to it, and of course gravity in $1+1$ or $2+1$ dimensions does not have any propagating degrees of freedom. Spacetime exterior to the defect is therefore locally flat, and the gravitational 'field' shows up only globally. For the string, this is the well-known conical deficit, and for the wall, the spacetime consists of two portions of flat spacetime (the interior of an accelerating bubble) glued across their boundaries. Viewed from the frame of the wall itself, the metric ceases to be static, and inertial observers appear to accelerate away from the wall. At the location of the 'core' of the string or wall, one finds, in the limit of vanishing thickness, a rather mild distributional singularity, either a finite discontinuity in the extrinsic curvature for the wall, or a simple apex of the cone spacetime.

Such global, long range effects of low codimension branes are highly restrictive if one wants to compactify the extra dimensions. This is unsurprising in the case of the domain wall, since the wall spacetime itself has compact spatial sections; 'standard' compactification with domain walls requires extra bulk fields [8], or walls with negative tension [3]. Compactification with cosmic strings on the other hand is also problematic, due to the global deficit angle. One can, in analogy with the wall, introduce a negative tension string for compactification on a torus, or indeed consider a global, rather than local, string [5, 6] which effectively introduces a bulk field in the guise of the Goldstone boson of the broken global symmetry. In addition however, one can consider compactifications on surfaces of positive total curvature using the deficit angle to provide the requisite closure [9]. However, the fact that no strong curvatures appear near the sources implies that the thin brane limit poses no problem, and hence the thickness of the brane can be neglected to a good approximation. The propagation of fields both in the bulk and on the brane is very weakly affected by whatever the internal microstructure of the brane may be. In other words, the low energy dynamics, in particular the gravity induced on the brane, is quite independent of any details about the core structure.

Branes of codimension $n = 3$, or higher, look in this respect qualitatively different. If their sources are well localized in the extra dimensions, then so is the gravitational field they produce: a localized object of codimension n creates a gravitational potential that falls off at large transverse distance as $1/r^{n-2}$. Then, sufficiently far away from the brane, the geometry of spacetime is hardly affected by the presence of the brane. So, with such branes, compactification of the extra dimensions presents no special difficulties, for as long as the thickness of the brane is sufficiently smaller than the size of the extra dimensions. On the other hand, the backreaction effects are now shifted to the region close to the brane. Since the source is localized, in the limit of zero thickness it can be expected to develop a naked singularity, where the curvature diverges (horizons surrounding the entire brane are to be avoided). This singularity will be smoothed out by the physical core of the brane, and the distance at which this happens depends on the specific details of the core model. This leaves open the door to the possibility that low energy physics acquires a strong dependence on the brane thickness, in contrast to what happened for thin domain walls and cosmic strings.

For a typical topological defect, such as, say, a cosmic string, the tension and the thickness are independent quantities, being determined by different combinations of the parameters of the Lagrangian (the Higgs self coupling and vev). So there is the possibility that the parameters that control the brane thickness and brane tension, call them r_b and r_0 respectively, take on very different values. Such an effect, which is mostly inconsequential for domain walls and strings, will allow higher codimension branes to generate a hierarchy between the fundamental scale and the scale of gravity on the brane. As we will see below, for r_b much smaller than r_0 the relation between the gravitational couplings in the bulk and on the brane takes the form

$$G_4 \simeq \frac{G_{4+n}}{V_0}\left(\frac{r_b}{r_0}\right)^{\frac{n-2}{2}\sqrt{\frac{n-1}{n+2}}}\,, \tag{1.1}$$

where V_0 is the volume of the extra dimensions, and we assume r_0 to be sufficiently smaller (though not necessarily very much smaller) than $V_0^{1/n}$.

Equation (1.1) differs from the standard result from Kaluza-Klein reduction by the factor involving the brane parameters r_0 and r_b, which is absent if the self-gravity of the brane is neglected. This factor is a consequence of the warping induced by the brane in its vicinity, and as we see, can have an important effect on the strength of gravity on the brane. An extreme possibility is that the volume of the extra dimensions be on the same scale as the fundamental scale (set by G_{4+n}) — say, around the TeV scale — and the small value of G_4 be attained by having r_b/r_0 small enough.

Hence, even if the starting point resembles the scenario of ADD in that we have a higher dimensional brane in a compact, empty bulk, the resolution (or better, reformulation) of the hierarchy problem in the scenario we have described is perhaps more akin to that proposed in RS1 [3]. As in RS1, the hierarchy is generated by the curvature of the spacetime near the brane, and not by a large internal volume. Also, the mass of the Kaluza-Klein states is set by $V_0^{-1/n}$. So, if the latter is on the TeV scale, the large multiplicity of very light Kaluza-Klein states that is typical of scenarios with large extra dimensions is avoided.

Note, however, some important differences (besides the different number of dimensions): in RS1 the hierarchy is generated by an exponential factor, whereas here we only have a power law. Hence, an “unnaturally” large number has to be introduced by hand. Explaining this factor requires a more fundamental theory (as happens with the large volume factor in the ADD scenario). Also, in this model there is only one kind of brane, one of positive tension, which can be regarded as a “TeV-brane.” Gravity becomes Planck-size away from the brane, where the warping becomes negligible. Finally, the way the warping of spacetime is generated involves in a crucial way the structure of the brane core, in particular its thickness. This, as far as we know, is a novel feature in brane-world scenarios.

Requiring r_b/r_0 to be small enough to generate the whole hierarchy between the TeV and Planck scales is perhaps an extreme situation — say, for an eleven dimensional universe, $n = 7$, a value $r_b/r_0 \sim 10^{-15}$ would be required. In particular, one has to deal with the very large curvatures induced near the brane. But, at any rate, the effect encoded in eq. (1.1) can have a significant effect in ADD scenarios, by altering the extra volume required to generate the hierarchy without changing the energies at which Kaluza-Klein states start to be excited, the effect being larger for higher values of n. One could envisage, e.g. having r_b on the fundamental scale, and r_0 being somewhat larger, which would enhance the hierarchy factor for a given volume.

The setup we have considered is a minimalistic one, designed to incorporate the effect of self-gravity of the brane-world into scenarios like the one of ADD with three or more extra dimensions. To this end, we consider a brane-world with the following features. First, we shall not specify any mechanism that gives rise to the brane. The brane will simply be a thin source (in the limit, a distributional source) of stress-energy, extended over three spatial directions. It is not clear whether field-theoretical topological defects can be used to model this — to date the only work on higher codimension brane worlds either uses global defects, [10], or is non-specific about the brane core, [11], indeed there is some dispute as to whether one can smooth the core without introducing long range effects [12]. Nonetheless, the solutions we use as a model for the self-gravitating higher codimension defect, [13], are qualitatively the same even in the presence of Coulomb fields, and so from the viewpoint of this paper, we regard this issue as being outside the regime of low energy physics. We shall content ourselves if the singularity can be smoothed, in principle, by a generic brane core. For simplicity, the only bulk field with non-vanishing background values will be taken to be the gravitational one, without a bulk cosmological constant. We shall also require the three-brane to admit a Poincaré-invariant ground state.

Some of these assumptions might need to be dropped for more fully realistic model-building, moreover, modification of some of these assumptions may give rise to qualitative changes. For example, branes in string theory include a variety of bulk fields which typically do not vanish for the background. In fact, no branes such as we require are known to exist in string theory, and its actual p-branes do not share many of the features we have discussed. Nevertheless, as we will argue near the end, the effect that gives rise to (1.1) is rather generic for branes. Hence we think it is important to study the consequences of this simplest brane-world model. The effects we describe, and the way we deal with the problems posed by the singularities are likely to be useful also in other settings.

One final aspect to be mentioned is that, in this framework, gravity is in no way localized near the brane. This is as in the conventional ADD scenario, and in order to recover four-dimensional gravity at large distances on the brane, we shall assume the extra dimensions to be compactified. No specific compactification scheme will be considered. It might be that a modification of the model allows for localization of gravity on the brane, e.g. by the inclusion of a cosmological constant, or otherwise. Actually, there have already been attempts at this. In [11], a search for self-gravitating, strictly local defects, such as we consider here, which yield finite gravity on the brane without compactification, resulted in a negative outcome. Addition of extra bulk fields appeared to remedy the situation. There are several points of contact between [11] and our analysis here, but the focus is somewhat different. Besides this, in [14, 15] a model was proposed with a brane in a flat Minkowski space with an arbitrary number of dimensions. It was claimed that the model achieved a strong localization of gravity. Self-gravity of the brane was neglected in that work, but a natural way of incorporating it would be along the lines described in this paper. More recently, other related recent work has appeared in [16].

2. Three-branes of codimension n

Our purpose is to describe three-brane geometries in $4+n$ dimensions. In the bulk of spacetime only gravity is present, which is described by the $4+n$ dimensional Einstein-Hilbert action. In addition, there will be a source of tension λ along the four brane-world coordinates x^μ. This source we take to be localized — as a first instance, we take it to be distributional.

Since we require that the brane admits a four-dimensional Poincaré invariant vacuum, for a single brane in a non-compact space we take the ansatz

$$ds^2 = A^2(r)\eta_{\mu\nu}dx^\mu dx^\nu + B^2(r)dr^2 + C^2(r)r^2 d\Omega_{n-1}^2\,, \tag{2.1}$$

with $\mu,\nu = 0,\ldots,3$.

The cases $n = 1, 2$ correspond to the well-known domain wall and local cosmic string geometries. For the domain wall, in an otherwise empty Minkowski space there are no Poincaré invariant umbilic surfaces ($K_{\mu\nu} \propto g_{\mu\nu}$) with nonzero extrinsic curvature, so walls with the above symmetries require a bulk negative cosmological constant or additional bulk fields. For $n = 2$, the solution is analogous to a cosmic string, which creates a conical deficit angle. In this case $A = B = C = 1$, and the transverse angle is identified with period $2\pi(1-4G_6\lambda)$.

Our interest lies in $n \geq 3$. For these cases, the equations have been solved in [13], with the result

$$A = f^{-\frac{1}{4}\sqrt{\frac{n-1}{n+2}}}\,, \qquad B = f^{-\frac{1}{n-2}\left(\frac{n-3}{2}-\sqrt{\frac{n-1}{n+2}}\right)}\,, \qquad C = f^{\frac{1}{n-2}\left(\frac{1}{2}+\sqrt{\frac{n-1}{n+2}}\right)}\,, \tag{2.2}$$

with

$$f = 1 + \left(\frac{r_0}{r}\right)^{n-2}\,. \tag{2.3}$$

For example, for the case $n = 3$, one gets $A = f^{-1/2\sqrt{10}}$, $B = f^{2/\sqrt{10}}$, and $C = f^{1/2+2/\sqrt{10}}$.

The parameter r_0 is directly related to the brane tension. The relation between them can be obtained by either reading the energy-momentum tensor of the brane from the Einstein equations near $r = 0$, or by a computation of the brane tension in an ADM-like approach [17], focusing on the asymptotic region in the direction transverse to the brane. In this way the stress tensor is computed to be $T_\mu^\nu = \lambda\delta_\mu^\nu$, with

$$\lambda = \sqrt{(n-1)(n+2)}\,\frac{\Omega_{n-1}}{32\pi G_{n+4}}\,r_0^{n-2}\,, \tag{2.4}$$

where Ω_{n-1} is the area of a unit $(n-1)-$sphere. Since the space in the directions transverse to the brane is empty, static and asymptotically flat, this has to be the same as the tension from the source at $r = 0$.

The non-analytic form of the metric coefficients is already an indication of the presence of singularities. Let us analyze them in more detail. First of all, the curvature grows to infinity as one approaches $r = 0$. Given the highly localized nature of the source, a naked singularity was indeed expected. This singularity we identify with the core of the brane. It was shown in [13] that, quite generally, an adequate (unspecified) choice of core model should smooth out the singularity. We will assume that smoothing out the core has the effect that the solution (2.2) is valid only down to a radius $r = r_b$, and that for $r < r_b$ the factors A, B, C take values not too different from their value at $r = r_b$. Hence, the metric induced on the brane is $A^2(r_b)\eta_{\mu\nu}dx^\mu dx^\nu$.

The space near $r = 0$ is highly distorted: notice that $A(r)$ shrinks to zero at $r = 0$. Since $A(r)$ measures the proper size in the directions parallel to the brane, we see that the brane at $r = 0$ has zero size! The singularity is at a finite proper radial distance, $\int_0 drB$. In contrast, the size of the spheres S^{n-1}, rC, diverge at $r = 0$. The latter implies a significant departure from the situation in flat space, where the transverse spheres S^{n-1} shrink to zero at the origin. In the present case, it means that the three-brane is actually delocalized over these spheres. This delocalization was present also in [11]. Note that the parameter r_b associated to the thickness of the brane is not a proper radius. As a matter of fact, having a small r_b does not necessarily imply that the brane is thin: if the thickness is measured by the 'area radius', $rC(r)$, then small r_b means that the brane is actually a thick one! Nevertheless, we will continue to refer to r_b as the 'brane thickness'.

JHEP05(2001)026

At large distances from the brane the geometry becomes asymptotically flat, as it typically does for localized objects of codimension $n \geq 3$. Hence, the extra dimensions can be compactified in much the same way as in the absence of the brane, if the compactification scale R_c is sufficiently larger than r_0 (or r_b). In that case, at distances less than R_c the geometry will be well approximated by (2.2). To take the compactification effects into account, in the following we will simply assume that the transverse radial distance is cutoff at a large distance, $r \leq R_c$. The ratio r_0/R_c may not need to be too large: one or two orders of magnitude may be enough.

We end this section by mentioning the relationship between the brane solutions of (2.2) and pointlike solutions to Einstein-scalar gravity in $(n+1)$ dimensions. If we compactify the spatial brane directions à la Kaluza-Klein, then the remaining spacetime would be of the form

$$ds^2 = -f^{\alpha} dt^2 + f^{\beta} dr^2 + r^2 f^{\gamma} d\Omega^2_{n-1} \tag{2.5}$$

(in Einstein conformal frame), where,

$$\alpha = -\frac{1}{2}\sqrt{\frac{n+2}{n-1}}\,,\ \beta = -\frac{1}{n-2}\left(n-3-\frac{1}{2}\sqrt{\frac{n+2}{n-1}}\right)\,,\ \gamma = \frac{1}{n-2}\left(1+\frac{1}{2}\sqrt{\frac{n+2}{n-1}}\right).$$

These are static, spherically symmetric solutions to $(n+1)$-dimensional gravity coupled to a scalar field, the scalar coming from the spatial volume of the three-brane. For example, for $n = 3$ the solutions describe the singular pointlike objects of [18] (see also [19] for a general discussion).

3. Propagation of scalars

The brane thickness and brane tension parameters, r_b and r_0, are in principle independent of each other, and fixed by the core model for the brane. We will be mostly interested in having $r_b \ll r_0$.

In this case, we have to show that one can consistently define the propagation of fields in this background for arbitrarily small values of r_b, in particular for $r_b \to 0$. In this respect, the problem is reminiscent of that of an electron in a Coulomb potential (the hydrogen atom): even if the source of the potential is a nucleus of finite size (the analog here of a brane of finite thickness), it is possible to solve for a singular pointlike source and obtain sensible results by discarding solutions that are badly behaved at the origin. The Sturm-Liouville boundary value problem is then well defined,[1] and unitarity is preserved. After this, the finite nucleus size can be treated as a small perturbation. Similarly here, the finite thickness of the brane will result in small corrections to the propagation of the field in the bulk.

[1] One needs also to impose suitable boundary conditions at infinity. We deal with this problem in a way different from the hydrogen atom, since we consider space to be compact.

JHEP05(2001)026

We shall study the propagation of a massless, minimally coupled scalar field Φ in the background of (2.2). Gravitons share many of the properties of these scalars, but also present important peculiarities of themselves, and will be the subject of the next section. Our analysis here will share some technical aspects with the ones in [5, 11], although the singularities in those cases lay away from the core of the brane.

The equation to study is then

$$\Box_{4+n}\Phi = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\Phi\right) = 0\,, \tag{3.1}$$

in the curved background of (2.2). Given the symmetries of the background, we separate variables as

$$\Phi = e^{ip_\mu x^\mu} Z_{lm_i}(\Omega)\phi_{ml}(r)\,, \tag{3.2}$$

where $p^2 = -m^2$ is the squared mass of the mode from the brane-world viewpoint, and $Z_{lm_i}(\Omega)$ are the spherical harmonics for the $(n-1)$-sphere,[2] with eigenvalue $l(l+n-2)$.

The equation (3.1) is simplified by noting that $\sqrt{-g} = A^4B(Cr)^{n-1}\omega_{n-1} = r^{n-1}C^2\omega_{n-1}$ (where ω_{n-1} is the measure on the unit $(n-1)$-sphere, which cancels), and $C^2/B^2 = f$. The scalar field equation for (3.2) then reduces to

$$-\frac{1}{r^{n-1}C^2}\left(r^{n-1}f\phi'_{ml}\right)' + \frac{l(l+n-2)}{r^2C^2}\phi_{ml} = \frac{m^2}{A^2}\phi_{ml}\,. \tag{3.3}$$

In order to study the qualitative features of (3.1) it is convenient to introduce the 'tortoise' coordinate

$$r_* = \int_0^r dr\frac{B}{A}\,. \tag{3.4}$$

The point $r = 0$ corresponds to $r_* = 0$, and asymptotically $r_* \sim r$. We also change to a new function $\psi_{ml}(r)$ (see [20]),

$$\psi_{ml}(r) = W(r)\phi_{ml}(r), \qquad W(r) \equiv (rC)^{\frac{n-1}{2}}A^{3/2}\,. \tag{3.5}$$

Doing so, we find a Schrödinger type of equation for ψ_{ml},

$$\left(-\frac{d^2}{dr_*^2} + \frac{1}{W}\frac{d^2W}{dr_*^2} + \frac{A^2}{r^2C^2}\,l(l+n-2)\right)\psi_{ml}(r) = m^2\psi_{ml}(r)\,. \tag{3.6}$$

The "effective potential" V for the field ψ is $\frac{1}{W}\frac{d^2W}{dr_*^2}$, which goes to $-\infty$ at $r_* = 0$, then reaches a maximum, and asymptotes to a constant value as $r_* \to \infty$ (see figure (1)). Near $r_* = 0$ the potential diverges, while A^2/C^2 goes to zero. Hence, all modes will behave near the singularity like the s-wave component of the massless

[2] Recall that, even if in higher dimensions there can be more than one independent angular momenta, in a spherically symmetric background they are degenerate and the wave equation depends only on a single number l.

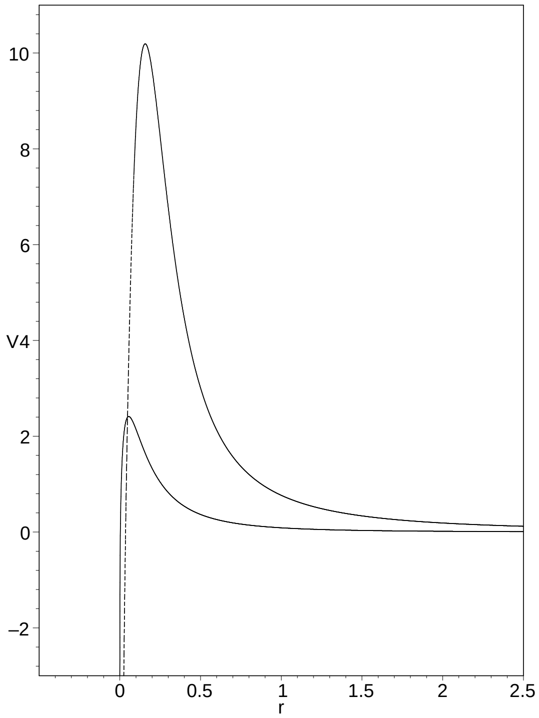

Figure 1: Plot of the potential $V = V(r)$ for $n = 3$ (solid line) and $n = 4$ with $r_0 = 0.5$

zero mode, $m = l = 0$. Notice that this is unusual for partial waves with $l > 0$, and is in contrast to motion in flat space, or in the Coulomb potential where, for $l > 0$, centrifugal barriers dominate at short distances. Here, however, the size of the transverse spheres, rC, grows large as $r \to 0$, so the "angular velocity" $\sim J/(rC)$ is small there.

Therefore, in order to investigate the effects of the singularity we need only to examine the s-wave of the zero mode, ϕ_{00}. It is easy to solve for it as

$$\psi_{00}(r_*) = aW(r_*) + bW(r_*)\log f\,, \tag{3.7}$$

this is,

$$\phi_{00}(r) = a + b\log f\,, \tag{3.8}$$

with integration constants a, b. Now the question is what boundary conditions we must impose at $r_* = 0$ in order to have a well defined boundary value problem. Elementary considerations for Sturm-Liouville problems dictate that the conditions are that $\phi_{00}(0)$ and $\phi'_{00}(0)$ be bounded. Hence we set $b = 0$, which implies that the s-wave of the zero-mode is a constant. The existence of this solution was clear from the very beginning, but what is less trivial is that all other modes will behave similarly near $r = 0$.

To normalize this s-wave zero mode we use the standard Sturm-Liouville measure in the normalization integral, with the weight function $w(r) = \omega_{n-1}r^{n-1}C^2/A^2$ (=

$\sqrt{-g}/A^2$) giving the value of a as

$$a^{-2} = \Omega_{n-1} \int dr \; r^{n-1} \frac{C^2}{A^2} \,. \tag{3.9}$$

This is convergent at its lower limit $r = 0$, hence we can get a finite amplitude even without the need to introduce a cutoff near $r = 0$. On the other hand, the upper bound $r \leq R_c$ is needed in order to render the amplitude a non-zero. This is not unexpected, since we anticipated the need to compactify the extra dimensions. To leading order in r_0/R_c we have

$$a^{-2} \simeq \Omega_{n-1} \frac{R_c^n}{n} \equiv V_0 \,, \tag{3.10}$$

where, to this order, V_0 is the volume of the extra directions. Note that the actual volume is different, $V = \Omega_{n-1} \int dr \; r^{n-1} C^2/A^4$. However, both the amplitude of the zero mode on the brane, a, and V are finite even including the singularity at $r = 0$, and to leading order for large R_c, $a^{-2} \simeq V \simeq V_0$.

The same boundary conditions can, and must, be imposed for all other modes. For the massless zero mode, the solution for the higher partial waves $l > 0$, and $n > 2$, can be found explicitly as Legendre functions, $\phi_{0l} \propto P_{\frac{l}{n-2}}(x)$ with $x = \frac{2r^{n-2}}{r_0^{n-2}} + 1$ (for $n = 3$ we get Legendre polynomials P_l). These modes take on non-zero values at $r = 0$, but they grow as r^l at large r, as in the absence of brane self-gravity. They will be eliminated once the boundary conditions for compactification are imposed.

Massive modes also tend to a finite value at $r = 0$. At larger values of r their radial wavefunctions asymptote to their flat space form in terms of Bessel functions, $\sim r^{1-n/2} J_{\frac{n+2l-2}{2}}(mr)$. Again, only the s-wave component is expected to remain after compactification. The shape of some of these modes is exhibited in figure (2).

With these results in hand, we are ready to study the interactions mediated by scalars, on either brane or bulk. The (euclidean) scalar two-point Green's function between points with brane and bulk coordinates $(x^\mu, \mathbf{r})$ and $(x^{\mu\prime}, \mathbf{r}')$, respectively, can be computed from the (normalized) eigenmodes ϕ_{ml} as

$$G(x, x'; \mathbf{r}, \mathbf{r}') = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu(x^\mu - x^{\mu\prime})} \sum_{m=0}^{\infty} \sum_{l,m_i} Z_{lm_i}(\Omega) Z_{lm_i}(\Omega') \frac{\phi_{ml}(r)\phi_{ml}(r')}{p^2 + m^2} \,. \tag{3.11}$$

Compactification will eliminate higher partial waves, and will also make the mass spectrum discrete. Then, for two points on the brane at a large distance, the effective interaction potential will be predominantly mediated by the zero mode, and be given by

$$\frac{\phi_{00}(0)^2}{|x - x'|} \simeq \frac{1}{V_0} \frac{1}{|x - x'|} \,. \tag{3.12}$$

Corrections to the leading order value $\phi_{00}^2 \simeq V_0^{-1}$ will appear at order $(r_0/R_c)^{n-2}$. The discrete massive modes will, as usual, give finite corrections of Yukawa type.

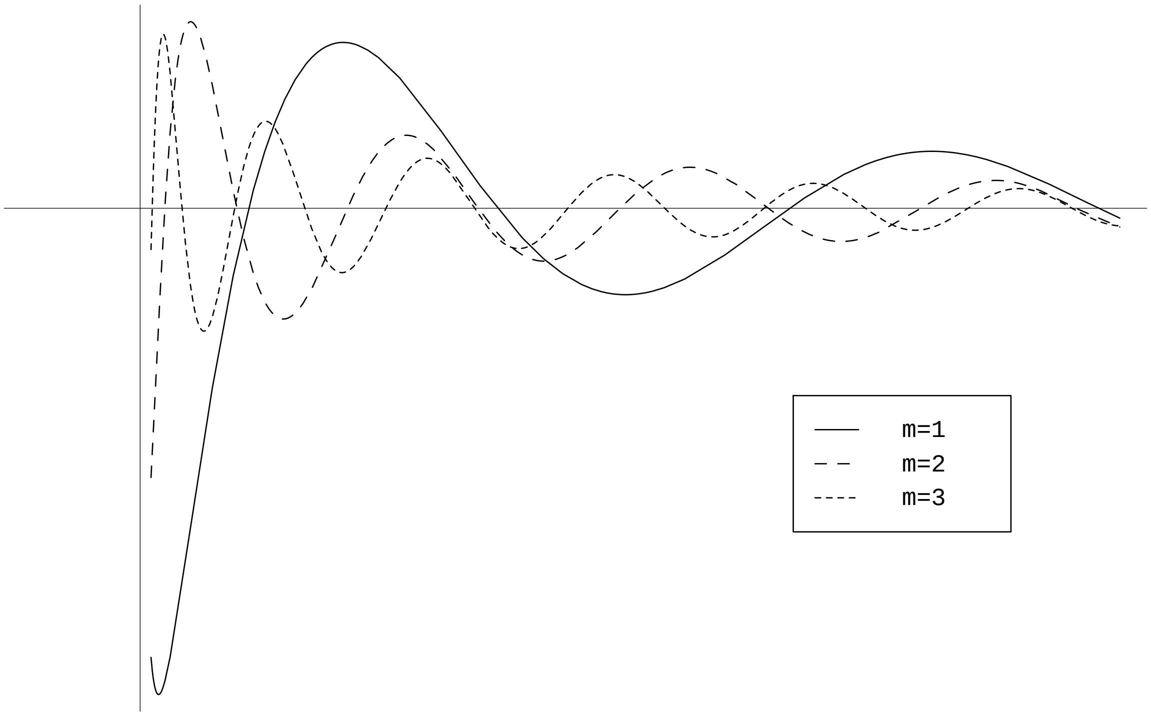

Figure 2: Plot of massive mode wavefunctions for $n = 3$, $r_0 = 1$.

Finally, we can also study the stability of the perturbations. The argument is a standard one. First, write (3.6) as

$$\left(\frac{d}{dr_*} + \frac{d\log W}{dr_*}\right)\left(-\frac{d}{dr_*} + \frac{d\log W}{dr_*}\right)\psi_{ml}(r) + \frac{A^2}{r^2C^2}l(l+n-2)\psi_{ml}(r) = m^2\psi_{ml}(r)\,. \tag{3.13}$$

With the boundary conditions specified above, the operators $\frac{d}{dr_*} + \frac{d\log W}{dr_*}$ and $-\frac{d}{dr_*} + \frac{d\log W}{dr_*}$ are each the adjoint of the other. Hence the operator that acts on ψ_{ml} on the left hand side of (3.6) is a positive semidefinite self-adjoint operator. Its eigenmodes then provide a complete basis, and their eigenvalues m^2, are all non-negative. These perturbations are, then, stable.

Hence we have proven that even if the background metric exhibits a naked singularity at the position of the brane, the propagation of minimal scalars, and the interactions mediated by them, on either brane or bulk, are well defined. The inclusion of a small but finite brane thickness cutoff r_b can then be treated as a small perturbation.

In this sense, we have shown that the solutions (2.1), (2.2) are the genuine cousins of the conical string, and distributional wall, spacetimes — in that the nominal singularity of the metric is quite integrable as far as causal propagation of fields on the spacetime is concerned.

4. Gravity on the brane

The leading static scalar potential (3.12) appears to be finite and largely independent of the cutoff imposed by the thickness of the brane.[3] However, notice that, in (3.12), the distance $|x - x'|$ is a coordinate distance, whereas the proper distance would be $A(r_b)|x - x'|$. This gets smaller as r_b decreases. Assuming, as we said, that on the brane the functions A, B, C do not vary appreciably from their value at $r = r_b$, then the proper distance between two points on the brane will be $\ell = A(r_b)|x - x'|$, and the leading order static interaction potential

$$V(x, x') = \frac{g_{4+n} A(r_b)}{V_0} \frac{1}{\ell}\,, \tag{4.1}$$

where g_{4+n} is the scalar coupling in $4 + n$ dimensions. Hence it appears that the effective coupling on the brane, g_4, takes the form

$$g_4 = \frac{g_{4+n} A(r_b)}{V_0}\,. \tag{4.2}$$

This is different from the standard result in the absence of brane self-gravity, which would not have the factor $A(r_b)$. As r_b gets smaller, the coupling on the brane gets weaker, and vanishes for a brane of zero thickness. Therefore, self-gravity makes the effective coupling induced on the brane acquire a strong dependence on the brane thickness.

Indeed, this feature would have been expected not only for scalars, but also for the gravity that is induced on the brane. Typically, in a brane world model, gravity becomes weaker where the brane metric is smaller, in the sense that the conformal factor A^2 in front of the metric becomes smaller.[4] In the present case, the proper size of the brane is proportional to $A(r_b)$, which decreases as r_b gets smaller.

Let us see in more detail how this comes about. To study the propagation of gravitons in the background of these branes, we perturb the metric $g_{ab} = g^0_{ab} + h_{ab}$, where g^0 is the background metric of (2.2), and we choose a gauge where perturbations are transverse and trace-free (TTF). The perturbations with indices along the brane metric, $h_{\mu\nu}$, will be related to tensor gravity on the brane. The brane scalars $h_{rr}, h_{\theta\theta}, h_{r\theta}$ give rise to massless scalars, which, upon compactification, correspond to the moduli for deformations of the internal space. These are a source of phenomenological difficulties, common to any compactification scheme. In the following, (in keeping with much of the literature) we shall assume that there exists a mechanism that gives a large mass to the moduli and stabilizes them up to some high energy

[3] Our views and interpretation in this section have been greatly improved by comments from Valery Rubakov.

[4] This is also familiar from string theory, where the supergravity solutions for fundamental strings or D-branes show that they can be coupled perturbatively (weakly) to gravity, i.e. to closed strings.

scale. If this is the case, they will decouple at low energies, so these components of h_{ab} can be set to zero. We will come back to this point in the final section.

With these assumptions, it is enough to perturb $A^2\eta_{\mu\nu} \to A^2\eta_{\mu\nu} + h_{\mu\nu}$ (in TTF gauge, $\nabla_\mu h^{\mu\nu} = h^\mu_\mu = 0$). Let us factor out the conformal factor and define $\hat{h}_{\mu\nu} = A^{-2}(r)h_{\mu\nu}$. The resulting Lichnerowicz equation is

$$\left[\Box_x \hat{h}_{\mu\nu} + 2R_{0\mu\lambda\nu\sigma}\hat{h}^{\lambda\sigma}\right] - \frac{A^2}{\sqrt{-g}}\left[\frac{\sqrt{-g}}{B^2}\hat{h}'_{\mu\nu}\right]' = 0\,, \tag{4.3}$$

where primes denote r-derivatives. The polarization structure can also be factored out, $\hat{h}_{\mu\nu}(x,z) = \epsilon_{\mu\nu}\Phi(x,z)$, with $\epsilon_{\mu\nu}$ a constant polarization tensor. Doing this, the equation that results for Φ is precisely the same as for a minimally coupled scalar, which we have studied in the previous section. Hence we find that, in a compact space, the interactions are dominated by the s-wave of the zero mode of the graviton. In order to determine the gravitational coupling on the brane it is then enough to focus on the latter mode.

In fact, the easiest way to do so is by working with the s-wave of the zero mode of the graviton exactly, i.e. beyond the linearized approximation. This is easy, since that mode is constant. Non-perturbatively, this is to say that the metric (2.2) continues to be a solution (with the same functions A, B, C, and f), when instead of the Minkowski metric we place an arbitrary Ricci-flat metric $\hat{\gamma}_{\mu\nu}(x)$,

$$ds^2 = A^2(r)\hat{\gamma}_{\mu\nu}(x)dx^\mu dx^\nu + B^2(r)dr^2 + C^2(r)r^2 d\Omega^2_{n-1}\,. \tag{4.4}$$

This $\hat{\gamma}_{\mu\nu}(x)$ is the non-perturbative counterpart of the s-wave of the zero mode of the perturbation $\hat{h}_{\mu\nu}$. Actually, the metric induced on the brane is not $\hat{\gamma}_{\mu\nu}(x)$, but instead $\gamma_{\mu\nu}(x) = A^2(r_b)\hat{\gamma}_{\mu\nu}(x)$, hence we write

$$ds^2 = \frac{A^2(r)}{A^2(r_b)}\gamma_{\mu\nu}(x)dx^\mu dx^\nu + B^2(r)dr^2 + C^2(r)r^2 d\Omega^2_{n-1}\,. \tag{4.5}$$

The (μ,ν) components of the Ricci tensor in the bulk $R_{\mu\nu}$, are equal to those of the Ricci tensor in the brane, $\mathcal{R}_{\mu\nu}(\gamma)$, so

$$R = \frac{A^2(r_b)}{A^2(r)}\mathcal{R}\,. \tag{4.6}$$

The relationship between the Newton constant in the bulk and in the brane follows easily now by integrating the extra dimensions in the Einstein-Hilbert action,

$$\frac{1}{G_{4+n}}\int dr\; d\Omega^{n-1}\; d^4x\sqrt{-g}R = \frac{1}{G_4}\int d^4x\sqrt{-\gamma}\mathcal{R} + \dots\,, \tag{4.7}$$

using (4.6),

$$\begin{aligned}\frac{G_{4+n}}{G_4} &= \int d\Omega^{n-1}dr\;\frac{A^2(r_b)\sqrt{-g}}{A^2(r)\sqrt{-\gamma}} = \frac{1}{A(r_b)^2}\int d\Omega^{n-1}dr\; r^{n-1}\;\frac{C^2}{A^2} \\ &\simeq \frac{V_0}{A^2(r_b)}\,,\end{aligned} \tag{4.8}$$

where, up to the constant $A^2(r_b)$, the integral is the same as appeared in eqs. (3.9) and (3.10), and is evaluated under the same approximations. For $r_b \ll r_0$ and using the expression for A in (2.2), we obtain the result we announced in the introduction (1.1),

$$G_4 \simeq \frac{G_{4+n}}{V_0} \left(\frac{r_b}{r_0}\right)^{\frac{n-2}{2}\sqrt{\frac{n-1}{n+2}}} . \tag{4.9}$$

This is our main result. It implies that the dimensionless ratio r_b/r_0 can be used for, or contribute to, generating a hierarchy between the fundamental scale of G_{4+n} and the Planck scale on the brane of G_4. There are several possibilities. One is that the whole hierarchy is generated by this ratio, with $R_c(\sim V_0^{1/n})$ being at the fundamental scale. In this case, the hierarchy problem consists of explaining why the brane thickness and tension have such an unnaturally small ratio.[5] Another possibility is that the hierarchy is partly generated by the volume factor, and partly by the brane parameters ratio. Since the curvature near the brane is controlled by r_b, the latter might be at the fundamental scale, while r_0 might be somewhat different. The interesting feature of this result is that the hierarchy can be changed without significantly varying the mass of the Kaluza-Klein states, which is determined by $1/R_c$.

Finally, it might perhaps be argued that the factor $(r_b/r_0)^{\frac{n-2}{2}\sqrt{\frac{n-1}{n+2}}}$ could be absorbed by a redefinition of the internal volume. However, we have seen that the actual internal volume is finite even for $r_b = 0$, and moreover, that would not be a useful viewpoint, since the mass of the Kaluza-Klein states is set by $V_0^{-1/n}$. The extra factor comes essentially from the curvature scalar R. Again, this is similar to the situation in the RS1 model.

5. Black branes and black holes on the brane

By exciting only the s-wave of the zero mode one can construct *black n-branes*. If the effect of compactification is neglected, they correspond to

$$ds^2 = A^2(r)\left[-\left(1-\frac{2m}{\rho}\right)dt^2 + \frac{d\rho^2}{1-\frac{2m}{\rho}} + \rho^2 d\Omega_2^2\right] + B^2(r)dr^2 + C^2(r)r^2 d\Omega_{n-1}^2 \tag{5.1}$$

(taking, again, (2.2) for A, B, C).[6] The size of the horizon in the directions transverse to the brane grows as $r = 0$ is approached, while in the parallel directions it shrinks. The two effects considered, the overall size grows, but the entropy remains finite: the

[5] I.e. small r_b/r_0, but we remind the reader that in a certain sense small r_b implies a thick brane.

[6] As a side remark, note that this solution describes a localized intersection of branes, i.e. the three-brane (the brane-world) is intersected by the black n-brane that extends in the directions transverse to the former. It is expected by no-hair arguments that such intersections are singular, as is so in this case.

area of the horizon of the black brane is

$$\mathcal{A} = \Omega_{n-1} \int dr\ r^{n-1} A^3 C^{n-1} = \Omega_{n-1} \int dr\ r^{n-1} \frac{C^2}{AB}\,, \tag{5.2}$$

and it is easy to see that even if the integrand diverges at $r = 0$, the divergence can be integrated. Hence, the entropy of these black branes is finite (if in a compact space) and it gets somewhat higher near the brane-world than it would be if the bulk were not curved by the brane-world.

This solution, like its counterparts in the absence of the 3-brane [21], or in anti-de Sitter space [22] is in fact unstable if the size of the horizon, $2m$, gets smaller than the compactification length. Recall that in [21], it was shown that a black string (metric (5.1) with $A = B = C = 1$) is unstable to fragmentation in the extra $\mathbf{r}_n$ dimensions. The instability takes the form of a 4D massive tensorial TTF s-wave perturbation:

$$h_{ab} = e^{i\mu\cdot r} \begin{bmatrix} \tilde{h}_{tt} & \tilde{h}_{t\rho} & 0 & 0 & \\ \tilde{h}_{t\rho} & \tilde{h}_{\rho\rho} & 0 & 0 & \mathbf{0} \\ 0 & 0 & \tilde{h}_{\theta\theta} & 0 & \\ 0 & 0 & 0 & \tilde{h}_{\theta\theta}\sin^2\theta & \\ & & & & \\ & & \mathbf{0} & & \mathbf{0} \end{bmatrix}. \tag{5.3}$$

The effect of this instability (due to the $e^{i\mu\cdot r}$) is to induce a rippling in the event horizon — which by entropic arguments we expect to fragment. The key feature of this instability (demonstrated and exploited in [22]) is that it lies purely within the 4D $\mu - \nu$ coordinates, and the action of the full Lichnerowicz operator produces a 4D Lichnerowicz operator with a mass term arising from the prefactor depending on the orthogonal directions

$$\Delta_{n+4} h_{ab} \propto \left(\Delta_4 + M^2\right) \tilde{h}_{\mu\nu}\,. \tag{5.4}$$

In the case of flat extra dimensions, this observation is trivial, however, if the extra dimensions are curved, as in (5.1) (or as in the RS scenario [22]) this splitting in fact still occurs. Therefore, the general instability will have the form $f_{\rm M}(\mathbf{x}_\perp)\tilde{h}_{\mu\nu}$, where $\tilde{h}_{\mu\nu}$ is the massive tensorial instability of (5.4) and $f_{\rm M}(\mathbf{x}_\perp)$, is the appropriate orthogonal eigenfunction giving rise to that mass. So, for the standard p-brane instability of [21], where the transverse dimensions are flat, the eigenfunction is $e^{i\mu_i x^i}$; for anti-de Sitter space, it is the appropriate Randall-Sundrum orthogonal eigenfunction $u_m(z)$ [4, 22]. In our case, we have demonstrated the eigenfunctions ϕ_{ml} in section 3, and therefore the appropriate form of the instability is

$$h_{\mu\nu} \sim A^2(r) Z_{lm_i}(\Omega)\phi_{ml}(r)\tilde{h}_{\mu\nu}\,. \tag{5.5}$$

The horizon therefore develops a instability, rippling with a shape given by the waveforms in figure 2.

The black brane in a large enough compact space is expected to collapse into a black hole. How to get an exact description of the black hole on the brane remains an open problem for $n \geq 3$, even when brane self-gravity is neglected.[7] In the latter approximation, and if the finite compactification size effects can be neglected, then the black hole is well approximated by the $4+n$ dimensional Schwarzschild solution, at least for distances on the brane that are smaller than the compactification length. In coordinates adapted to the presence of the brane, this is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + r^2 d\Omega_{n-1}^2 + \frac{2m}{(r^2 + x^i x^i)^{(n-2)/2}} dt^2 - \frac{2m}{(r^2 + x^i x^i)^{n/2}} (r dr + x^i dx^i)^2 \tag{5.6}$$

$(i = 1, 2, 3)$. The brane corresponds to the section $r = 0$.

Brane self-gravity will significantly distort this geometry near the core of the brane. In particular, we have seen that the size of the spheres S^{n-1} grows larger near the brane, and the directions parallel to the brane to shrink. Hence, we can expect that the black hole will pinch at the brane core in directions parallel to the brane, but will greatly expand in the directions transverse to it. As was the case for the black brane, these two competing effects are expected to balance in the calculation of the area of the horizon in the bulk, which is expected to remain finite. In general, as argued in [24], black holes remain attached to the brane. In the present case this may be even more so, since the area of the horizons gets comparatively higher near the brane.

6. Discussion

Let us now discuss, mostly from a qualitative point of view, an issue we have left aside so far, namely the compactification moduli. Even before dealing with their stabilization, we should expect the moduli amplitudes to be very large at the position of the brane. It was already observed in [25] that in the RS1 model, the corresponding modulus (the radion) gets large there where gravitons parallel to the brane get small, and vice versa. This is, in fact, a consequence of Einstein's equations in vacuo. In our case, it would be in accord with the fact that the metric in the directions transverse to the brane grows without bound as r_b gets smaller, while it goes to zero in the parallel directions.

Now suppose that we compactify the extra dimensions and somehow manage to stabilize the moduli and give them a large mass. The infinite (for $r_b = 0$) amplitude of the moduli at the brane would seem to imply that the brane is extremely sensitive to (scalar) deformations of the compact manifold. In order to avoid this, the compact manifold has to be very rigid, i.e. the mass of the moduli has to be very large. With finite brane thickness, the minimum mass of the moduli can be expected to be determined by, roughly, $1/r_b$. Since the equations for parallel gravitons and moduli

[7] Toy models for lower dimensional branes are more tractable, see e.g. [23].

do not decouple, any fluctuations in the moduli will have an effect on the parallel gravitons. Therefore, the question of moduli stabilization is actually of relevance to the entire setup. Nevertheless, we have seen that if it can be appropriately solved, then all other issues about the consequences of brane self-gravity and the influence of the brane thickness can be safely dealt with.

Although we have been considering the specific case of what is arguably the simplest three-brane solution, the weakening of gravity on the brane is actually rather generic. Recall that the warp that gives rise to the factor r_b/r_0 in (1.1) is due to the shrinkage of the metric along the directions parallel to the brane, while the brane core remains at a finite proper spatial distance. This shrinkage is, in fact, a generic consequence of the tension along the brane directions. Hence, once one has a brane with a certain positive tension, the same shrinkage will happen even in the presence of extra bulk fields, for as long as the gravity of the latter is not so strong as to upset the effect. In fact, the non-generic cases will be those for which the extra bulk fields act in such a way as to exactly counterbalance the shrinkage of the geometry — such is the case of e.g. D3-branes, for which the gravity of the RR field has the effect of pushing the brane core down an infinite throat.

To conclude, we have shown how to incorporate the gravitational field induced by a brane with three or more transverse dimensions, and have found that it can lead to important effects. Although the solutions (2.1), (2.2) exhibit singularities at the core of the brane, the singularities are such that bulk fields can propagate in their presence in a finite and unitary way. Hence, we can legitimately regard these solutions as the higher dimensional analogues of the domain wall and cosmic string spacetimes. Concerning the gravity that is induced on the brane, a new dimensionless parameter, which is determined by the brane structure, enters into the determination of the effective four dimensional gravitational constant. This can be used to generate, or contribute to, a hierarchy between the Planck scale and a fundamental scale. Nevertheless, the energy scale for Kaluza-Klein excitations remains largely unaffected by variations of that parameter. Such an influence of brane parameters on the hierarchy is a novel effect, to be added to ADD scenarios. It mixes in the main ingredient present in the RS1 model, namely, the warping created by the brane self-gravity, and hence provides the possibility of taking advantage of some of the features of each model.

Of course, if one wishes instead to deal with a genuinely fat brane-world, one must first find a reasonable core model. Ghergetta et al. [11] point out some problems with a fully localised core, and it is quite possible that none such exists. If one wishes to model the brane by a topological defect (as in [10]) then one generally has a spontaneously broken symmetry in the vacuum of the theory. For a defect of codimension n, this mandates a nontrivial Π_{n-1} homotopy group of the vacuum manifold (or coset space G/H). For simple Lie groups, this is equivalent to $\Pi_{n-2}(H)$, the unbroken group, but if H is nontrivial, then there will be Coulomb fields surrounding the core of the defect [12] — in other words, we might expect any smooth brane-world *not* to have a fully localised core. Another possibility is that the brane is not a field-theoretical defect, but a brane originating from string theory. It cannot be a D3-brane, but the latter is in fact a very special case due to conformal invariance. More realistic stringy brane-worlds may well exhibit the kind of warping we have described.

Indeed, string theory suggests a way in which a small ratio between brane thickness and brane tension can arise. Suppose that string theory, or whatever the fundamental theory is, has branes of the form we require (or similar, in that they present the appropriate shrinkage in brane directions while keeping the brane core at a finite proper distance). For a single brane in that theory the tension and thickness scales would both be at the fundamental scale, say the string scale l_s. Now, if instead of one brane we have a stack of N branes, then the thickness would presumably still be $r_b \sim l_s$, but since we have N branes the tension would have grown. In the absence of interactions between the branes one would have, if eq. (2.4) holds, $r_0 \sim N^{1/(n-2)} l_s$. Interactions betwen the branes, if attractive, will reduce this value, but at any rate the tension will grow as N grows. Therefore r_0/r_b grows with N, and a large hierarchy appears for large N. A number of issues has to be dealt with, since having a stack with a large number of branes increases accordingly the number of brane fields, including the light ones. A way to prevent them from interfering with the fields we actually observe would have to be provided. The question of whether this is a workable direction remains an open one.

For now, we cannot fully answer all these questions in detail, but we simply note that we have shown that the brane-world gravity can be dramatically dependent on the specific model used to generate the core.

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A. How to obtain a "Planck brane"

If the effect of brane self-gravity is used to generate the entire hierarchy between the TeV and Planck scales, then, in the terminology of [3] the brane would be a "TeV-brane." In contrast to [3], there is no "Planck-brane" in this model. A "probe" Planck brane would have to sit at a location where the metric factor $A(r)$ takes on values of order unity.

Nevertheless, there is a way to make the self-gravitating brane of (2.2) a Planck brane. Indeed, it allows to find finite gravity on the brane even in the limit $r_b \to 0$, as occurs for the domain wall or cosmic string brane-worlds. In that case, the limit where the brane has zero size is sensible on its own, and the physics at low energies is actually independent of any details of the brane structure. This might be a desirable feature.

The procedure to achieve this lies on an alternative, if somewhat *ad hoc* definition of what the brane metric is, i.e. to which metric does matter on the brane couple to. Usually, fields that live on the brane are just a truncation of fields that actually can extend into the bulk, but whose propagation into the bulk is heavily suppressed, by say, the topology of the field configuration. This is, they are actually bulk fields, and hence they will couple to gravity through the metric induced on the brane. Now, let us suppose instead that the fields that live on the brane do not couple to the metric induced on the brane (which has zero size if $r_b = 0$), but rather to one that is related by a constant rescaling. This implies rescaling the strength of the gravitational interactions on the brane, which can be done in such a way that a finite brane metric results, and finite gravity on it, even if $r_b \to 0$. More specifically, in the spacetime (4.4)

$$ds^2 = A^2(r)\hat{\gamma}_{\mu\nu}(x, r, \Omega)dx^\mu dx^\nu + B^2(r)dr^2 + C^2(r)r^2 d\Omega^2_{n-1}\,, \qquad \text{(A.1)}$$

let us *define* our brane metric to be $\hat{\gamma}_{\mu\nu}(x) \equiv \hat{\gamma}_{\mu\nu}(x, r = 0)$. By this we mean that matter fields on the brane will couple to gravity through the metric $\hat{\gamma}_{\mu\nu}$. For example, we take the action for a brane-scalar φ as

$$\frac{1}{2}\int d^4x\sqrt{-\hat{\gamma}}(\hat{\gamma}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \mu^2\varphi^2)\,. \qquad \text{(A.2)}$$

Equivalently, in the linearized approximation where $\hat{\gamma}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}$, the coupling to a brane source of stress-energy $T_{\mu\nu}$ will be through a term

$$\frac{1}{2}\int d^4x T_{\mu\nu}\hat{h}^{\mu\nu}\,. \qquad \text{(A.3)}$$

This is different to coupling to the actual metric perturbation $h_{\mu\nu} = A^2(r)\hat{h}_{\mu\nu}$. In fact, a coupling of the form (A.3) does not arise naturally in any of the common ways to generate brane fields from a topological defect. However, one may argue that the question of how to generate such couplings belongs in the realm of the higher theory that gives rise to the brane, and which we have explicitly left out of our discussion.

It is now a straightforward matter to show, following the steps that lead to (1.1), that to leading order in r_0/R_c, the Newton constant on the brane is given by the standard formula,

$$G_4 \simeq \frac{G_{4+n}}{V_0}\,, \qquad \text{(A.4)}$$

instead of (1.1). Notice that even if $\hat{h}_{\mu\nu}$ satisfies the same equation as the scalar Φ of the previous section, the argument that the distance $|x - x'|$ that appeared in the interaction potential (3.12) is not the proper distance is not relevant anymore. For, if brane matter couples to the metric $\hat{\gamma}_{\mu\nu}$, then $|x-x'|$ is the distance actually measured by a brane observer (with rulers made of brane matter), and the interaction potential is indeed of the form (3.12), without any extra factors.

We can now analyze the different gravitational interactions between brane and bulk matter. We have seen that $\hat{h}_{\mu\nu}$ satisfies the same equation as the minimal scalar Φ of the previous section. Hence, for two sources on the brane, which couple to gravity through the interaction term (A.3), the relevant two-point function is

$$\langle \hat{h}_{\mu\nu}(x)\hat{h}_{\rho\sigma}(x')\rangle = \epsilon_{\mu\nu}\epsilon'_{\rho\sigma}G(x,x';0,0)\,, \tag{A.5}$$

where $G(x,x';0,0)$ is that of (3.11) with the two points at $r=0$. This is finite, and all the comments made about the scalar interaction on the brane through (3.11), apply here as well.

Now, if one of the sources is on the brane and the other is in the bulk, with the stress-energy tensor of the latter lying along the brane directions, and coupling as $\sim \int T_{\mu\nu}h^{\mu\nu} \sim \int A^2(r)T_{\mu\nu}\hat{h}^{\mu\nu}$, then

$$\langle h_{\mu\nu}(x,z)\hat{h}_{\rho\sigma}(x',0)\rangle = \epsilon_{\mu\nu}\epsilon'_{\rho\sigma}A^2(r)G(x,x';z,0)\,. \tag{A.6}$$

Again, this is finite, although it becomes smaller as the bulk source gets closer to the brane.

This serves us to illustrate one of the important features of this prescription for obtaining a Planck brane, namely, that brane matter and bulk matter are, with this prescription, fundamentally different. The gravitational interactions of matter in the bulk (at least those mediated by gravitons with polarization parallel to the brane) become weaker closer to the brane. At $r=0$, its coupling to gravity vanishes. In contrast, brane matter gravitates finitely, but if one tried to pull it out of the brane its weight would presumably become infinite. Hence such brane matter would have to be permanently confined to the brane. Bizarre as this may sound, the prescription appears to be a consistent one, for which the issue of the correct coupling of gravity can only be resolved from the point of view of the theory that underlies the brane structure. The most striking consequence of this is that finite results can be obtained even if the proper size of the brane has shrunk to zero!

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